Analysis of a self-excited two-mass system

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A simple two-mass self-excited system is analysed. The stability of the equilibrium position is investigated. Two alternatives of self-excitation are considered – to the upper mass or to the lower mass. The passive and active suppression means are investigated. The active suppression means are due to the periodic variation of the upper mass.

Key words: self-excited vibration, passive and active suppression means, parametric excitation by mass variation

1. Introduction

A two-mass chain self-excited system is considered characterized by the upper mass m_1 connected with the mass m_2 by a spring having stiffness k_1; mass m_2 mounts have the stiffness k_2 (see Fig. 1). A question arises whether such a simple model can give the sufficient information on a real system, e.g. high structure induced by flow. In most cases the self-excited vibration of these structures corresponds mostly to the first or second vibration mode and so on a simplified model different effects can be analysed. When these first vibration modes and natural frequencies are known it is possible to build corresponding simple model especially when one- or two-mass concentrations exist.

The simple two-mass models have been analysed in several publications (see e.g. in [1] where only passive suppression means are considered). The passive and active suppression means for this two-mass model are analysed in [2], where the upper mass is self-excited and the active means are due to the parametric excitation given by stiffness variation of the lower mass mounts.

Two alternatives of self-excitation are considered in this contribution: Alternative I: mass m_1 is self-excited (Fig. 1a), Alternative II: mass m_2 is self-excited (Fig. 1b). (The subsystem with upper mass m_1 can be considered as a tuned absorber.) For both alternatives upper mass m_1 is varied.

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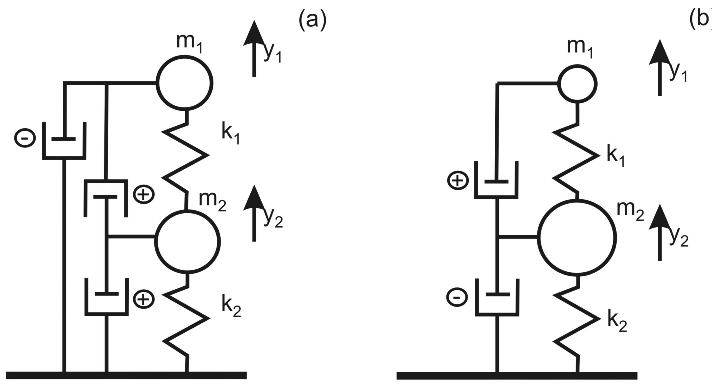

Fig. 1. Scheme of the system: a) Alternative I, b) Alternative II.

The lateral deflections are denoted by y_1, y_2. In most real systems like high structures the deflections are horizontal, i.e. Fig. 1 is rather schematic. The damping and self-excitation are expressed by linear terms depending on y_1' or y_2'. We shall suppose that the damping and self-excitation are expressed by linear viscous damping (of course, the self-excitation by the linear viscous damping with the negative coefficient), and the equilibrium position is given by the trivial solution ($y_1 = y_2 = 0$) of the differential equations of motion. Then we can limit the stability analysis on the linearized differential equations of motion because the aim of the analysis is to stabilize the equilibrium position.

In this contribution both the passive as well as the active suppression means will be analysed. The active means are given by the periodic variation of the upper mass: $m_1 = m_{10}(1 + e\cos\omega t) = m_{10}(1 + \varepsilon e_0 \cos\omega t)$. The parametric excitation as the means for self-excited vibration suppression using mass variation was analysed first in [3] where a general analysis is presented and illustrated by an example of a two-mass system.

Also the upper mass was varied there and so we can utilize some results presented and make a deeper analysis. The main result presented in [3] as an example is as follows: The full suppression of vibration can be achieved at parametric excitation frequency given by the difference of natural frequencies of the abbreviated system (i.e. without small terms of the differential equations), when certain conditions are met.

Let us mention a paper dealing with parametric excitation using mass variation as the means for self-excited vibration suppression of the system with two degrees of freedom where the motion is given by lateral and angular deflections (see [4]).

It will be useful for the further analysis to mention the general results presented in the paper [3] (see also the references there, especially [5]), which will be presented in the next chapter.

2. General results

Let us consider a mechanical system with n degrees of freedom and n masses where only one mass is varied (the mass is expressed by the constant and harmonic component). The governing differential equations after transformation: $\omega_1 t = \tau$, ($\omega_1 = \sqrt{k_1/m_{10}}$) and in the quasi-normal form, not considering the non-linear terms, read:

$$y_s'' + \Omega_s^2 y_s = \varepsilon \left[-\sum_{k=1}^{n} \Theta_{sk} y_k' + e_0 \left(\sum_{k=1}^{n} \eta \sin \eta\tau \sum_{k=1}^{n} \vartheta_{sk} y_k' + \cos \eta\tau \sum_{k=1}^{n} Q_{sk} y_k \right) \right]. \tag{1}$$

For our two-mass system Eq. (1) has the form (see [3]):

$$y_s'' + \Omega_s^2 y_s = \varepsilon(\alpha_{s1} F_1 + \alpha_{s2} F_2), \qquad (s = 1, 2), \tag{2}$$

$$\begin{aligned} F_1 &= e_0 \cos \eta\tau \left[(1 - a_1) y_1 + (1 - a_2) y_2 \right] + e_0 \eta \sin \eta\tau (y_1' + y_2') \\ &\quad - \ (\kappa_1 + \kappa_{12}) \left[(1 - a_1) y_1' + (1 - a_2) y_2' \right], \\ F_2 &= (\kappa_1 + \kappa_{12}) M \left[(1 - a_1) y_1' + (1 - a_2) y_2' \right] - \kappa_2 (a_1 y_1' + a_2 y_2'), \\ \eta &= \frac{\omega}{\omega_1}. \end{aligned}$$

Corresponding coefficients are in the Appendix.

The conditions for stabilizing the equilibrium position given by trivial solution are formulated in [3]. Let us suppose that the system is unstable only for the j-th normal vibration mode, i.e. only Θ_{jj} is negative. The stability conditions of equilibrium position for $\eta_0 = |\Theta_j + \Theta_r|$ are:

$$\Theta_{jj} + \Theta_{rr} > 0, \tag{3}$$

$$e_0^2 \left(\eta_0 \vartheta_{jr} - \frac{Q_{jr}}{\Omega_r} \right) \left(\eta_0 \vartheta_{rj} - \frac{Q_{rj}}{\Omega_j} \right) - \Theta_{jj} \Theta_{rr} > 0. \tag{4}$$

Similarly for $\eta_0 = |\Theta_j - \Theta_r|$:
The first condition is equivalent with (3) and the second one is:

$$\left(\frac{Q_{jr}}{\Omega_r} - \eta_0 \vartheta_{jr} \right) \left(\eta_0 \vartheta_{rj} + \frac{Q_{rj}}{\Omega_j} \right) + \Theta_{jj} \Theta_{rr} > 0. \tag{5}$$

For the considered two-mass system the condition (4) is not met and so for active suppression it means that only the parametric excitation frequency $\eta_0 = \Omega_2 - \Omega_1$ will be considered.

There is an example in paper [3] where for the considered system with two alternatives the condition (5) after rearranging reads:

$$e_0^2 \alpha_{11} \alpha_{21} \Omega_1 \Omega_2 + \Theta_{11} \Theta_{22} > 0. \tag{6}$$

3. Alternative I

Let us start with the passive means. The equilibrium position will be stable when both coefficients Θ_{11}, Θ_{22} will be positive. Using the results in Appendix ((A.13)), coefficients Θ_{11}, Θ_{22} (taking for $\kappa_1 = -\beta$ to distinguish excitation) are:

$$\begin{aligned} \Theta_{11} &= -\alpha_{11}\beta + K_{11}\kappa_{12} + \alpha_{21}\kappa_2, \\ \Theta_{22} &= -\alpha_{21}\beta + K_{22}\kappa_{12} + \alpha_{11}\kappa_2. \end{aligned} \tag{7}$$

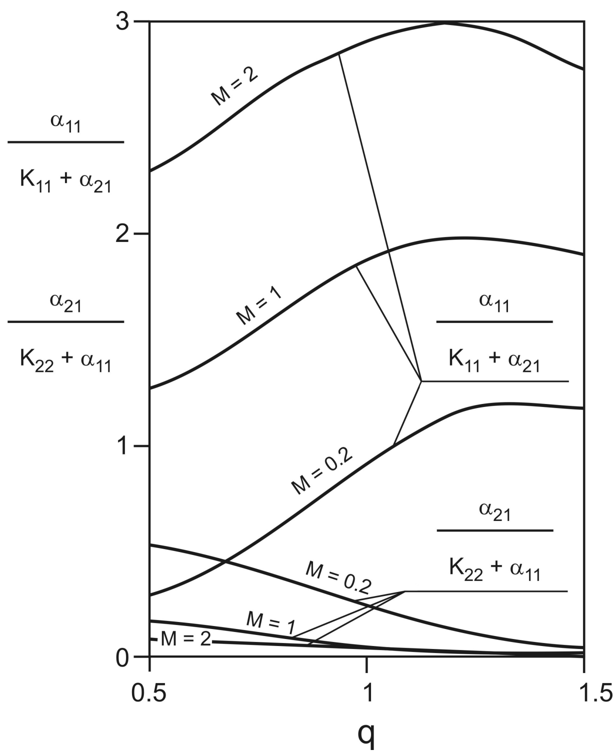

Fig. 2. $\frac{\alpha_{11}}{K_{11}+\alpha_{21}}$, $\frac{\alpha_{21}}{K_{22}+\alpha_{11}}$ in dependence on the tuning coefficient q for different values of mass ratio M.

For this alternative the following will be considered: $\kappa_{12} = \kappa_2 = \kappa$. Then the stability conditions are:

$$\kappa > \left[\frac{\alpha_{11}}{K_{11} + \alpha_{21}}\right]\beta, \qquad \kappa \geq \left[\frac{\alpha_{21}}{K_{22} + \alpha_{11}}\right]\beta. \tag{8}$$

A suitable case would be when both conditions are identical, i.e.:

$$\left[\frac{\alpha_{11}}{K_{11} + \alpha_{21}}\right] = \left[\frac{\alpha_{21}}{K_{22} + \alpha_{11}}\right]. \tag{9}$$

For illustration Fig. 2 shows dependences of $\left[\frac{\alpha_{11}}{K_{11}+\alpha_{21}}\right]$ and $\left[\frac{\alpha_{21}}{K_{22}+\alpha_{11}}\right]$ on the tuning coefficient q for three values of $M(0.2, 1, 2)$ and for $\beta = 0.03$, $\kappa_{12} = \kappa_2 = 0.02$. Intersection point of both curves exists only for the smallest M. So it is not always possible to fulfil condition (9). In this case the combination of passive and active means is suitable, as it will be shown further.

The parametric excitation of mass m_1 is varied, i.e. $m_1 = m_{10}(1+\varepsilon e_0 \cos\omega t)$. The conditions (3) and (6) must be met to stabilize the equilibrium position at $\eta_0 = \Omega_2 - \Omega_1$. Condition (3) for our system reads:

$$-(\alpha_{11} + \alpha_{21})\beta + (K_{11} + K_{22} + \alpha_{11} + \alpha_{21})\kappa > 0. \tag{10}$$

This is met for the given values of β and κ.

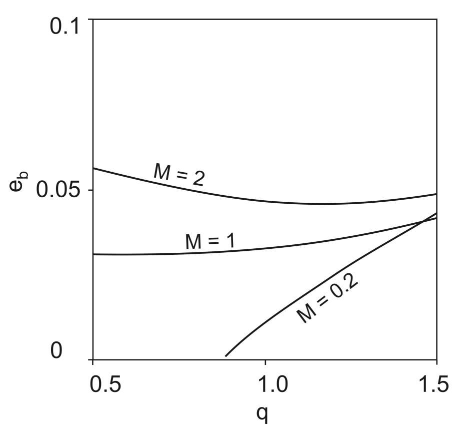

Fig. 3. Boundary value of e (denoted as e_b) in dependence on q for different M.

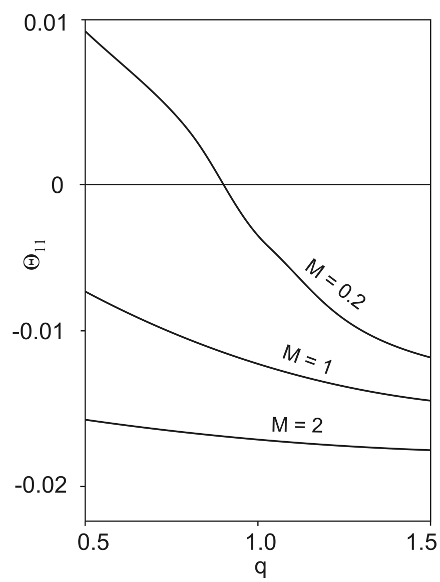

Fig. 4. Coefficient Θ_{11} in dependence on q for different M.

Condition (6) has the form:

$$\varepsilon^2\alpha_{11}\alpha_{21}\Omega_1\Omega_2 + \Theta_{11}\Theta_{22} = \varepsilon^2\alpha_{11}\alpha_{21}q + [-\alpha_{11}\beta + (K_{11}+\alpha_{21})\kappa][-\alpha_{21}\beta + (K_{22}+\alpha_{11})\kappa] > 0. \tag{11}$$

Condition (6), taken into account real eccentricity $e = \varepsilon e_0$, has the form:

$$\varepsilon^2\alpha_{11}\alpha_{21}q + \varepsilon^2[-\alpha_{11}\beta + (K_{11}+\alpha_{21})\kappa][-\alpha_{21}\beta\kappa + (K_{22}+\alpha_{11})\kappa] > 0. \tag{11a}$$

The boundary value of ε can be obtained from (11a) when replacing $>$ by $=$. This boundary value of e (denoted as e_{b}) can be obtained from equation:

$$e_{\mathrm{b}}^2 = \varepsilon^2\frac{[-\alpha_{11}\beta + (K_{11}+\alpha_{21})\kappa][-\alpha_{21}\beta + (K_{22}+\alpha_{11})\kappa]}{\alpha_{11}\alpha_{21}q}. \tag{12}$$

Figure 3 shows ε_{b} in dependence on q for different values of M and the same coefficients as in Fig. 2. When $e > e_{\mathrm{b}}$ then the equilibrium can be stabilized in broader interval of parametric excitation frequency. For further information, Fig. 4 shows Θ_{11} in dependence on q for different M. For $q < 0.8$ Θ_{11} is positive, which means that passive means are sufficient because Θ_{22} is positive in the whole range of q for the considered M.

4. Alternative II

Again first the passive means will be considered. The equilibrium position is stable when both Θ_{11}, Θ_{22} are positive:

$$\Theta_{11} = K_{11}\kappa_{12} - \alpha_{21}\beta, \qquad \Theta_{22} = K_{22}\kappa_{12} - \alpha_{11}\beta. \tag{13}$$

Again for the passive means only it seems that for vibration modes the stability boundary would be equal, i.e. $\Theta_{11} = \Theta_{22}$ which means:

$$\frac{\alpha_{21}}{K_{11}} = \frac{\alpha_{11}}{K_{22}}. \tag{14}$$

Figure 5 shows these parameters in dependence on tuning coefficient q for three values of $M(0.05, 0.1, 0.2)$ and for $\varepsilon\kappa_{12} = \varepsilon\kappa = 0.02 = \varepsilon\beta$. It can be seen that for all considered alternatives of M the intersection points exist. But if the stability condition is not met then this tuning is not suitable for application of parametric excitation because the necessary condition (3) would not be met. For

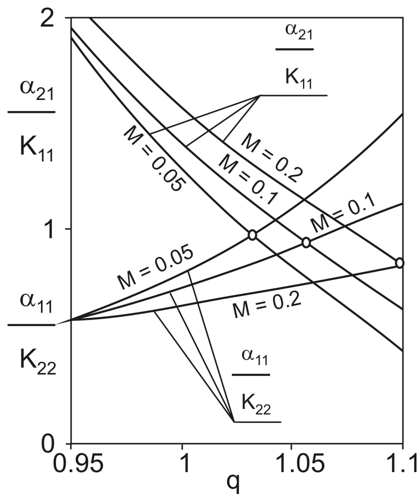

Fig. 5. $\frac{\alpha_{21}}{K_{11}}$, $\frac{\alpha_{11}}{K_{22}}$ in dependence on q for different M.

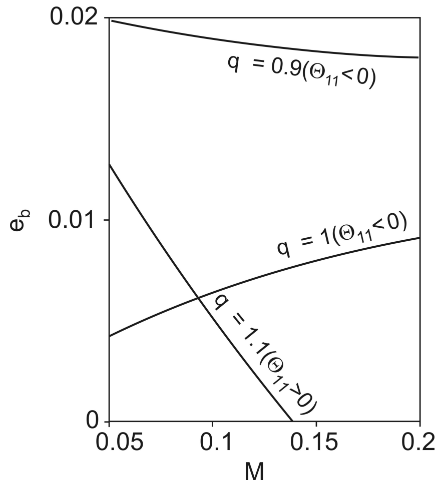

Fig. 6. The dependence of e_{b} on M for different q.

successful suppression by parametric excitation only a single vibration mode can be unstable before parametric excitation application (see also [7]).

For this alternative the boundary value e_{b} is presented as a function of M for three values of q $(0.9, 1, 1.1)$ – see Fig. 6. Each curve is marked whether Θ_{11} is negative/positive.

5. Conclusion

A simple two-mass self-excited system is analysed. As linear self-excitation due to the negative linear viscous damping is considered the stability of the equilibrium position can be determined by analysis of the linear differential equations of disturbances, which are the linearized differential equations of motion. These equations are transformed into the quasi-normal form.

Two alternatives of self-excitation (upper or lower mass) are analysed.

For the passive means (expressed by positive linear damping) the boundary values are determined and illustrated in Fig. 2 (Alternative I) and Fig. 5 (Alternative II). The condition for the case when stability conditions for both vibration modes merge into a single one is determined but this is not always possible to realize especially when the mass ratio M is not small. For this case by the combination of passive and active means a full suppression can be achieved.

The analysis results for the active suppression means by parametric excitation due to the upper mass variation are presented and illustrated in Figs. 3, 4 and 6.

The variation of the mass can be realized in real system, e.g. by rotating gears with unbalanced masses or crankshaft mechanism. By its constant rotation the total reduced mass is varied periodically. The suppression effect occurs at parametric excitation frequency $\eta_0 \cong \Omega_2 - \Omega_1$ and not at $\eta_0 \cong \Omega_1 + \Omega_2$, which is convenient by using mass variation because high rotation frequency is not necessary.

Appendix

Appendix I in the monograph [6] deals with the linear transformation of the two-mass chain system (see Fig. 1) into the quasi-normal form. This system without damping is governed by the following equations:

$$\begin{aligned} m_1\ddot{y}_1 + k_1(y_1 - y_2) &= 0, \\ m_2\ddot{y}_2 - k_1(y_1 - y_2) + k_2 y_2 &= 0. \end{aligned} \tag{A.1}$$

By the transformation $\omega_1 t = \tau$, where $\omega_1 = \sqrt{k_1/m_1}$, Eqs. (A.1) get the form:

$$\begin{gathered} y_1'' + y_1 - y_2 = 0, \\ y_2'' - M(y_1 - y_2) + q^2 y_2 = 0, \\ M = \frac{m_1}{m_2}, \quad q^2 = \frac{\omega_0^2}{\omega_1^2}, \quad \omega_0^2 = \frac{k_2}{m_2}, \quad \omega_2^2 = \frac{k_1 + k_2}{m_2}. \end{gathered} \tag{A.2}$$

The natural frequencies of the system governed by Eqs. (A.2) are:

$$\Omega_{1,2} = \left\{ \frac{1}{2}(1 + M + q^2) \pm \left[\frac{1}{4}(1 + M + q^2)^2 - q^2 \right]^{1/2} \right\}^{1/2}. \tag{A.3}$$

Applying transformation

$$\begin{aligned} y_1 &= u_1 + u_2, \\ y_2 &= a_1 u_1 + a_2 u_2, \end{aligned} \tag{A.4}$$

on Eqs. (A.2), they get the form:

$$\begin{aligned} u_1'' + \Omega_1^2 u_1 &= 0, \\ u_2'' + \Omega_2^2 u_2 &= 0. \end{aligned} \tag{A.5}$$

The following relations define coefficients a_1, a_2:

$$a_1 = M(q^2 + M - \Omega_1^2)^{-1},$$
$$a_2 = M(q^2 + M - \Omega_2^2)^{-1}. \tag{A.6}$$

The inverse transformation (A.4) reads

$$u_1 = \alpha_{11}y_1 + \alpha_{12}y_2,$$
$$u_2 = \alpha_{21}y_1 + \alpha_{22}y_2. \tag{A.7}$$

The coefficients α_{11}, α_{12}, α_{21}, α_{22} are defined as follows:

$$\alpha_{11} = \frac{a_2}{a_2 - a_1}, \quad \alpha_{12} = \frac{1}{a_1 - a_2}, \quad \alpha_{21} = \frac{a_1}{a_1 - a_2}, \quad \alpha_{22} = -\alpha_{12}. \tag{A.8}$$

Some further relations read:

$$\begin{gathered} 0 \le a_1 \le 1, \qquad a_2 \le 0, \\ a_1 a_2 = -M, \\ (1 - a_j) = \Omega_j^2, \qquad (j = 1, 2), \\ a_1 = a_2 = 1 - M - q^2, \\ \Omega_1\Omega_2 = q. \end{gathered} \tag{A.9}$$

If we take into account the linear viscous damping b_1, b_{12}, b_2, the two-mass chain system is governed by the following equations:

$$m_1\ddot{y}_1 + b_1\dot{y}_1 + b_{12}(\dot{y}_1 - \dot{y}_2) + k_1(y_1 - y_2) = 0,$$
$$m_2 - b_{12}(\dot{y}_1 - \dot{y}_2) - k_1(y_1 - y_2) + b_2\dot{y}_2 + k_2y_2 = 0. \tag{A.10}$$

Using the same time transformation as for (A.1) we obtain:

$$y_1'' + \kappa_1 y_1' + \kappa_{12}(y_1' - y_2') + y_1 - y_2 = 0,$$
$$y_2'' - M[\kappa_{12}(y_1' - y_2') - y_1 - y_2] + \kappa_2 y_2' + q^2 y_2 = 0. \tag{A.11}$$

After applying transformation (A.4) the following equations are obtained:

$$
\begin{aligned}
u_1'' + \Omega_1^2 u_1 + \Theta_{11} u_1' + \Theta_{12} u_2' &= 0,\\
u_2'' + \Omega_2^2 u_2 + \Theta_{21} u_1' + \Theta_{22} u_2' &= 0.
\end{aligned} \tag{A.12}
$$

The following relations are for the coefficients Θ_{jk}, $(j, k = 1, 2)$:

$$
\begin{aligned}
\Theta_{11} &= \alpha_{11}\kappa_1 + K_{11}\kappa_{12} + \alpha_{21}\kappa_2,\\
\Theta_{12} &= \alpha_{11}\kappa_1 + K_{12}\kappa_{12} - \alpha_{11}\kappa_2,\\
\Theta_{21} &= \alpha_{21}\kappa_1 + K_{21}\kappa_{12} - \alpha_{21}\kappa_{12},\\
\Theta_{22} &= \alpha_{21}\kappa_1 + K_{22}\kappa_{12} + \alpha_{11}\kappa_2,
\end{aligned} \tag{A.13}
$$

$$
K_{11} = \alpha_{11}(1 - a_1)^2, \qquad K_{12} = \alpha_{11}(1 - a_1)(1 - a_2),
$$
$$
K_{21} = \alpha_{21}(1 - a_1)(1 - a_2), \qquad K_{22} = \alpha_{21}(1 - a_2)^2.
$$

The graphs and tables of important coefficients in dependence on the tuning coefficient q for different values of mass M are presented in [6].

Acknowledgements

In memory of my dear friends Academician K. V. Frolov and Prof. I. Ballo.

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Received: 28.4.2008
Revised: 24.6.2008